$\leqslant NY$)

k = temperature dependent slag thermal conductivity

NX = number of physical cells in x'

NY = number of physical cells in y'

p = slag pressure

Pr = slag Prandtl number

$r(x')$ = distance of vessel wall from symmetry axis

Re = $\rho u\ \delta y/\mu$ = Reynolds number for the flow layer

S_a = $-\alpha_a(x)\ v_a(x)$ = source term from slag impinging on the boundary layer

t = time

t' = t

$T(x,y,t)$ = slag temperature

T_a = temperature of ash impinging on slag

T_f = slag freezing temperature below which there is no flow

T_g = reactor gas temperature

T_{sb} = $T[x, y_b(x,t), t]$

u = component of $\vec{v}(x,y,t)$ in the x-direction

$u_b(x)$ = velocity of the boundary in the x-direction

v = component of $\vec{v}(x,y,t)$ the the y-direction

$\vec{v}(x,y,t)$ = slag velocity

$v_a(x)$ = ash velocity normal to the slag layer

$v_b(x)$ = velocity of the boundary in the y-direction

x = coordinate denoting arc length along the wall in a vertical plane passing through the combustion chamber axis of symmetry

x' = x

y = coordinate normal to the wall in a vertical plane through the combustion chamber symmetry axis

$y_b(x,t)$ = boundary layer thickness

y' = $y/y_b(x,t)$

Greek Letters

α_a = ash volume fraction in gas adjacent to slag

δx = scale length for variation in x

δy = scale length for variation in y

ϵ = slag emissivity, ($\epsilon \simeq 0.5$)

$\theta(x)$ = angle between the local direction of the x-axis and the vertical

μ = temperature dependent dynamic viscosity

ν = temperature dependent slag kinematic viscosity

ρ = slag density

ρ_1 = ratio of convective to viscous terms in Eq. 1

ρ_2 = ratio of thermal convective to thermal diffusive terms in Eq. (4)

σ = Stefan-Boltzmann constant

ϕ = azimuthal reactor angle

∇ = the nabla or del operator (Bird et al., 1960)

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Manuscript received July 10, 1980; revision received October 20, and accepted October 24, 1980.

Heat Transfer to Laminar In-Tube Flow of Pseudoplastic Fluids

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This note considers the problem of heat transfer to laminar flow of pseudoplastic fluids in circular tubes with constant heat flux. The constitutive relation for pseudoplastic fluids is $\tau = K(\partial u/\partial y)^n$, with $n \leq 1.0$. Analytical solutions are available for constant property, fully developed flow, e.g., Grigull (1956). Numerical solutions have been presented for developing flow for selected values of n, e.g., McKillop (1964). In both regions, pseudoplastic fluids have higher heat transfer coefficients than Newtonian fluids. Mizushina et al. (1967), Cochrane (1969), and Mahalingam et al. (1975) report analyses which indicate that the temperature dependence of K further increases heat transfer coefficients for heating; however, their predictions are not sufficient for a general correlation.

Results of an efficient explicit numerical solution are presented here. A correlation of these results includes non-Newtonian and temperature-dependent K effects for entrance and fully developed regions.

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0001-1541-81-4814-0872-$2.00.

PROBLEM FORMULATION AND SOLUTION

In formulating the equations, the following assumptions were made:

1. The flow is steady and axisymmetric.
2. Axial conduction is negligible.
3. Free convection effects are negligible.
4. The usual boundary layer approximations are valid since pseudoplastic fluids exhibit flat velocity and temperature profiles near the tube centerline and sharp profile gradients near the wall.
5. K is temperature-dependent according to the constitutive equation of most industrial fluids:

$$K = a \exp[-bt] \qquad (1)$$

With these assumptions, the general governing equations in cylindrical coordinates (Figure 1) reduce to

Momentum (x-direction):

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{dp}{dx} + \frac{1}{r}\frac{\partial}{\partial y}(r\tau) \qquad (2)$$

Energy:

$$\rho u c_p \frac{\partial t}{\partial x} + \rho v c_p \frac{\partial t}{\partial y} = \frac{\partial}{\partial y}\left(rk\frac{\partial t}{\partial r}\right) + \mu_a\left(\frac{\partial u}{\partial y}\right)^2 \quad (3)$$

Continuity:

$$\frac{\partial}{\partial x}(\rho u r) + \frac{\partial}{\partial y}(\rho v r) = 0 \quad (4)$$

In these equations,

$$\tau = K\left(\frac{\partial u}{\partial y}\right)^n \text{ with } (n \leq 1) \quad (5)$$

and

$$\mu_a = K\left(\frac{\partial u}{\partial y}\right)^{n-1} \quad (6)$$

The global continuity equation is also applicable:

$$\int_A \rho u dA = \text{constant} \quad (7)$$

Far away from the tube inlet, fully developed velocity and temperature profiles exist. In this region, $v = 0$, $\partial u/\partial x = 0$, and $\partial t/\partial x = dt/dx$. With these simplifications, Eqs. 2 to 7 reduce to the fully developed governing equations.

The boundary conditions for this problem are for the velocity profile:

$$u(x, 0) = 0, \; v(x, 0) = 0, \; \left(\frac{\partial u}{\partial y}\right)_c = 0$$

for the temperature profile:

$$\left(k\frac{\partial t}{\partial y}\right)_w = q_w'' \text{ and } \left(\frac{\partial t}{\partial y}\right)_c = 0$$

The boundary conditions at tube inlet are

$$u(0, y) = u(y); \quad t(0, y) = t_0$$

$$p(0) = p_0$$

Thus, there are four equations, Eqs. 2, 3, 4 and 7, with four unknowns, u, v, t, and p, and there are sufficient boundary conditions. Therefore, this problem is mathematically well-posed.

It is seen from Eq. 6 that near the tube centerline, the apparent viscosity, μ_a, will be unrealistically high. To overcome this singularity, the viscosity is computed from the wall up to the edge of the boundary layer ($u/u_c < 0.995$). All intermediate points from the boundary layer edge to the centerline are assigned the same μ_a value.

The numerical method employed is described in detail in Joshi (1978) and Joshi and Bergles (1978, 1980). The procedure is explicit, utilizing a form of DuFort-Frankel (1953) differencing which results in a scheme for which the axial step size is not severely constrained due to stability considerations. A key element in the procedure is the determination of the axial pressure gradient for each axial step. This is accomplished by numerically integrating the finite-difference form of the axial momentum equation over the tube cross-section and employing the overall conservation of mass constraint to eliminate the integral of the axial velocity from the equation. The pressure gradient can then be evaluated explicity.

RESULTS

Constant-Property Solutions

In this analysis, predictions for the local Nusselt numbers were obtained for the entire thermal length, for $n = 1.0$, 0.75, 0.5 and 0.25. The numerical predictions for $n = 1$ were correlated by the method suggested by Churchill and Usagi (1972) as

$$Nu_{cp,n=1} = 4.36[1 + (0.376(X^+)^{-0.33})^6]^{1/6} \quad (8)$$

This correlation is in excellent agreement with the correlation developed by Churchill and Ozoe (1973). The present prediction, for all values of n and for the entire thermal length, are well correlated by the non-Newtonian correction, $\Delta^{1/3}$, suggested by

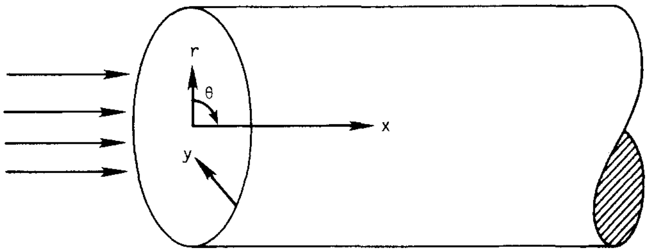

Figure 1. Cylindrical coordinate system.

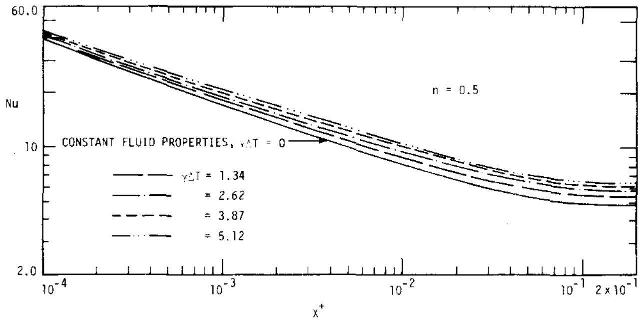

Figure 2. Dependence of Nusselt number on dimensionless distance and *consistency parameter*.

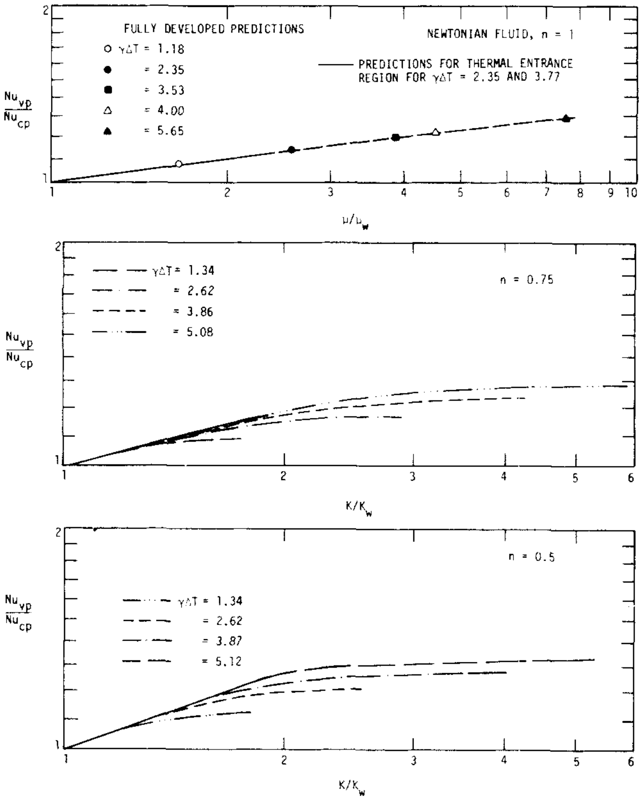

Figure 3. Dependence of Nusselt number ratio on ratio of viscosities or consistency indices.

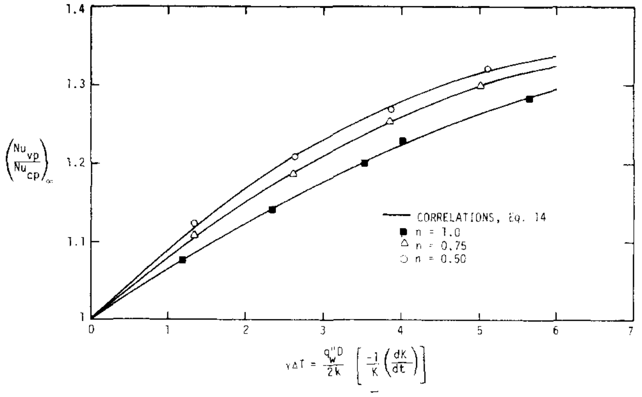

Figure 4. Dependence of Nusselt ratio on $\gamma\Delta T$.

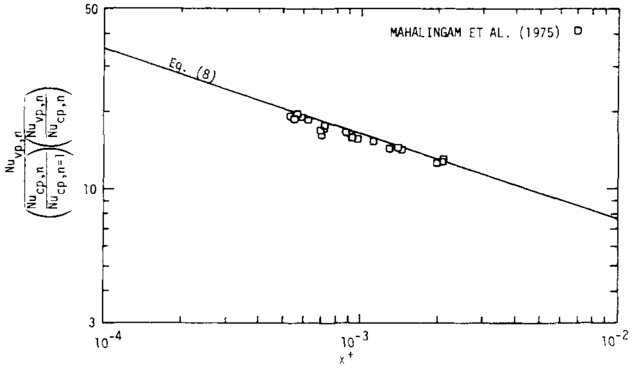

Figure 5. Comparison of numerical and experimental results.

Mizushina et al. (1967) for fully developed flow. Thus

$$\frac{Nu_{cp,n}}{Nu_{cp,n=1}} = \Delta^{1/3} \tag{9}$$

With Re as a generalized Reynolds number, the $f \cdot Re$ product for various n values was constant, equal to 16, which is identical to the exact analytical result.

In general, the above results confirm the accuracy of the explicit, finite-difference scheme.

Variable-Property Solutions

The variable-property solutions established the influence of the temperature dependence of K. The dimensionless parameter $\gamma\Delta T$ was introduced to account for temperature-dependent consistency. For liquids obeying the exponential relation of Eq. 1, $\gamma = b$; therefore,

$$\gamma\Delta T = b\frac{q_w''D}{2k} \tag{10}$$

In the present analysis, $\gamma\Delta T$ was varied by changing the wall heat flux. The solutions were obtained for four different heat fluxes for $n = 0.75$ and 0.5. For $n = 1.0$, the solutions were obtained for two different heat fluxes. These solutions were found to be sufficient for correlation. The value of b was chosen from the experimental data given for pseudoplastic fluids by Mahalingam et al. (1975) and Joshi (1978).

The variable property predictions for $n = 0.5$ are shown in Figure 2. There is a substantial increase in the Nusselt number above the constant property values. In order to correlate this effect for each n at constant $\gamma\Delta T$ and at several axial locations, a ratio of (Nu_{vp}/Nu_{cp}) was computed and plotted against the corresponding μ/μ_w or K/K_w. These results are shown in Figure 3. For the Newtonian case alone, an additional fully developed analysis was done, the results of which are also plotted in Figure 3. For a given wall heat flux, (K/K_w) is low near the tube entrance, while in the fully developed region, higher values of (K/K_w) exist. It is seen from Figure 3 that

1. For a Newtonian fluid for the entire length.a log-linear relationship is observed. The slope of this line is 0.14; therefore

$$\left(\frac{Nu_{vp}}{Nu_{cp}}\right)_{n=1} = \left(\frac{\mu}{\mu_w}\right)^{0.14} \tag{11}$$

This relation is in approximate agreement with the results of Yang (1962) who obtained the viscosity correction index $m = 0.11$.

2. For $n < 1$, a log-linear relation is observed in the thermal entrance region, while in the fully developed region an asymptotic behavior is observed. In the thermal entrance region, the increase in heat transfer coefficient is independent of $\gamma\Delta T$, while in the fully developed region the increase is independent of K/K_w. The general functional relationship for the Nusselt number is

$$Nu_{vp,n} = f[n, \gamma\Delta T, (K/K_w), X^+] \tag{12}$$

TABLE 1. PREDICTIONS OF m IN THE THERMAL ENTRANCE REGION

n	m
1.0	0.14
0.75	0.25
0.50	0.36

Variable Consistency Correlation

The correlation strategy was to correlate various property effects in the entrance region and in the fully developed region. An interpolation formula was then devised to correlate both regions with a single equation. The following general relation is proposed for the thermal entrance region:

$$\left(\frac{Nu_{vp,n}}{Nu_{cp,n}}\right)_e = \left(\frac{K}{K_w}\right)^m \tag{13}$$

where $m = 0.58\text{-}0.44n$ according to the data given in Table 1.

For the fully developed region, the plot of $(Nu_{vp}/Nu_{cp})_\infty$ against $\gamma\Delta T$ is shown in Figure 4 for $n = 1.0$, 0.75, and 0.5. These data are correlated as

$$\left(\frac{Nu_{vp,n}}{Nu_{cp,n}}\right)_\infty = 1 + (0.12392 - 0.0542\,n)\gamma\Delta T - (0.010133 - 0.0068\,n)(\gamma\Delta T)^2 \tag{14}$$

This equation is valid up to $\gamma\Delta T = 6$, which covers the range of normally encountered heat fluxes.

The two asymptotic correlations, Eqs. 13 and 14, were combined by the interpolation technique suggested by Churchill and Usagi (1972) as

$$Nu_{vp,n} = Nu_{cp,n=1}\frac{(Nu_{cp,n}/Nu_{cp,n=1})}{\dfrac{1}{(Nu_{vp,n}/Nu_{cp,n})_\infty}\left\{\left[\dfrac{(Nu_{vp,n}/Nu_{cp,n})_\infty}{(Nu_{vp,n}/Nu_{cp,n})_e}\right]^{30} + 1\right\}^{1/30}} \quad \text{for } n < 1 \tag{15}$$

where the necessary Nusselt numbers are given by Eqs. 8, 9, 13, and 14. The limiting case of $n = 1$ is given simply by Eq. 11. Equation 15 represents an accurate, explicit correlation of the numerical predictions, which is very convenient for design purposes.

COMPARISON WITH AVAILABLE DATA

Mahalingam et al. (1975) have reported one of the few experimental studies that have conditions similar to those assumed in this analysis. They used an electrically heated test section and studied a methocel solution, with consistency closely approximated by Eq. 1. Their data, when adjusted for non-Newtonian and temperature-dependent consistency effects according to Eq. 15, agree very well with the curve for $Nu_{cp,n=1}$, as shown in Figure 5.

It is interesting to note that Mahalingam et al. correlated their data as follows:

$$Nu = 1.46\left(\frac{K}{K_w}\right)^{0.14} \Delta_w^{1/3}[Gz + 0.0083(GrPr)_w^{0.75}]^{1/3} \tag{16}$$

In this correlation, the variable property effects are divided between temperature-dependent consistency index and density effects, in contrast to the present analysis which considers only temperature-dependent consistency index. However, it does appear that the analytical assumption of constant density is acceptable for the data of Mahalingam et al. since the free convection term for their data is very small (less than 5%). An extension of the present analysis would be of interest to confirm the influence of free convection effects, particularly for $n \simeq 1$, where large free convection effects are expected based on experience for pure water.

No experimental data are available in the fully developed region and in the transition region from the thermal entrance region to the fully developed region. Data are needed in these regions to examine fully the accuracy of the present correlation.

ACKNOWLEDGMENTS

This study was supported by the Engineering Research Institute, the Power Affiliate Program, and the Department of Mechanical Engineering at Iowa State University. Dr. R. H. Pletcher provided many helpful suggestions on the procedure for the finite-difference solution.

NOTATION

a, b = constants
A = tube surface area
c_p = isobaric specific heat
D = inside tube diameter
g = gravitational acceleration
h = heat transfer coefficient
K = consistency index
k = thermal conductivity
L = tube length
m = exponent
$\dot{m}$ = mass flow rate
n = flow behavior index (dimensionless)
p = pressure
Δp = pressure drop
q'' = heat flux
r = radial coordinate
ΔT = $q_w''D/2k$
t = temperature
Δt = temperature difference between wall and bulk
u = axial velocity
$\bar{u}$ = average axial velocity
v = radial velocity
x = distance from the tube inlet
y = distance from the tube wall

Greek Letters

β = isobaric coefficient of thermal expansion
γ = $-1/\mu\ d\mu/dt$ or $-1/K\ dK/dt$
Δ = $(3n + 1)/4n$
ρ = density
μ = viscosity
μ_a = apparent viscosity, Eq. 6
μ_{eff} = $\tau_w/(8\bar{u}/D)$
τ = shear stress

Dimensionless Numbers

f = Fanning friction factor $(\Delta pD/2L\rho u^2)$
Gr = Grashof number $(\rho^2 g\beta\Delta tD^3/\mu_{eff}^2)$
Gz = Graetz number $(\dot{m}c_p/kL)$
Nu = Nusselt number (hD/k)
Pr = effective Prandtl number $(\mu_{eff}c_p/k)$
Re = effective Reynolds number $(\rho D\bar{u}/\mu_{eff})$
X^+ = dimensionless distance $[2(x/D)/RePr]$

Subscripts

c = tube centerline
cp = constant properties
e = entrance region
o = tube entrance
vp = variable properties
w = wall temperature
∞ = fully developed condition

All fluid properties are evaluated at the bulk temperature unless otherwise noted.

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Manuscript received May 16, 1980; revision received August 19, and accepted October 15, 1980.